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Damping described by fading memory—analysis and application to fractional derivative models

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Abstract

Some damping models where the actual stress does not depend on the actual strain but also on the entire strain history are studied. Basic requirements in the frequency and time domain significant for the choice of damping model are outlined. A one-dimensional linear constitutive viscoelastic equation is considered. Three different equivalent constitutive equations describing the viscoelastic model are presented. The constitutive relation on the convolution integral form is studied in particular. A closed form expression for the memory kernel corresponding to the fractional derivative model of viscoelasticity is given. The memory kernel is examined with respect to its regularity and asymptotic behavior. The memory kernel's relation to the fractional derivative operator is discussed in particular and the fractional derivative of the convolution term is derived. The fractional derivative model is also given by two coupled equations using an ''internal variable''. The inclusion of the fractional derivative constitutive equation in the equations of motion for a viscoelastic structure is discussed. We suggest a formulation of the structural equations that involves the convolution integral description of the fractional derivative model of viscoelasticity. This form is shown to possess several mathematical advantages compared to an often used formulation that involves a fractional derivative operator form of constitutive relation. An efficient time discretization algorithm, based on Newmark's method, for solving the structural equations is presented and some numerical examples are given. A simplification of the fractional derivative of the memory kernel, derived in the present study, is then employed, which avoids the actual evaluation of the memory kernel.

1. Introduction

Perfectly elastic behavior of a material is an idealization; in reality in-elasticity is always present even at very low strains. This in-elasticity leads to energy dissipation or (material) damping. In the present study the energy dissipation is assumed to occur within the continuum element. The material is assumed to have a viscoelastic constitutive relation between stress and strain.

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Material damping is commonly quantified by loss factor η which is the ratio between energy lost and maximum energy stored within the continuum element under steady-state harmonic conditions. For linear materials the loss factor can be expressed as the ratio between the imaginary part and the real part of the complex modulus of elasticity in the frequency domain. Material damping is in the engineering community commonly approximated by a frequency independent modulus of elasticity in the frequency domain (i.e., by a constant loss factor). This constitutive model has no reasonable representation in the time domain. When extrapolated to the entire frequency domain this model represents a non-causal relation between stress and strain (i.e., it allows response before excitation, see Crandall, 1970). This unphysical behavior is overlooked in a fixed frequency analysis but when studying energy propagation phenomena and transient analysis this behavior is believed to be crucial. Even in a fixed frequency analysis it might be important to use a physically correct model. Another model often employed for approximating hysteretic damping is the so-called viscous damping. In this model the energy losses are proportional to the excitation frequency and the imaginary part of the complex modulus of elasticity is thus proportional to the excitation frequency. This model can be represented by the Kelvin model in the time domain where the stress is taken as the sum of parts proportional to the strain and the strain rate. The viscous model is often used due to its simplicity and mathematical convenience. Unfortunately this model is not appropriate for describing a wide class of nearly elastic engineering materials. For example, there is no instantaneous response upon a sudden change in stress, and the resulting loss factor is proportional to the excitation frequency. Engineering materials in general do not show this frequency dependence in their measured loss factors. Instead, it is found experimentally that many engineering materials show loss factors that are almost independent of excitation frequency over several frequency decades, see Kimball and Lovell (1927). Paradoxically, this resembles more the unphysical model referred to above.

In the present study we focus on some of the principles that should guide our choice of damping model. The basic requirements that should be imposed on any constitutive model are (see Bowen, 1989): consistency with fundamental axioms of mass, energy and momentum balance, as well as thermodynamic laws, coordinate and frame invariance, spatial locality and causality. A further requirement which concerns damping is the dissipativity condition (i.e., that energy should be dissipated from the system rather than introduced).

A special class of constitutive models having a fading memory, i.e., the stress that produces a strain does not only depend on the current strain but also on the previous strain history in such manner that the current stress depends stronger on the recent strain history than on the distant strain history, is studied in particular. The linear constitutive equation can be formulated on integral, hereditary or convolution form. The model is due to Boltzmann (1876). The regularity conditions that should apply for the kernel function in the hereditary model are discussed below. The hereditary model is, by its kernel or memory function, related to viscoelastic models such as that recently used by Lesieutre (1992), and the fractional derivative model of viscoelasticity, see Bagley and Torvik (1983). The fractional derivative model of viscoelasticity is formulated on integral form and a closed form expression for the corresponding memory function is given. How to include the fractional derivative model of viscoelasticity into structural equations of motion for a viscoelastic structure is discussed. We prefer to use a formulation of the structural equations that involves the constitutive equation of viscoelasticity on integral form. Finally, an algorithm for solving the structural equations of motion for a discretized fractional calculus viscoelastic system is given and some examples for a one-degree of freedom system are presented.

2. Constitutive equations of linear viscoelasticity

The one-dimensional linear constitutive equation of viscoelasticity in the time domain on differential operator form is often written as

$$\sum_{k=0}^{m} p_k \frac{\mathrm{d}^k \sigma(t)}{\mathrm{d}t^k} = \sum_{k=0}^{m} q_k \frac{\mathrm{d}^k \varepsilon(t)}{\mathrm{d}t^k} \tag{1}$$

where $\sigma(t)$ denotes the stress, $\varepsilon(t)$ is the strain and p_k and q_k are material parameters. Here, as in the whole present study, isothermal conditions are assumed and only uniaxial stress states are considered. Note that this form represents a non-unique relation between stress and strain, since there is a need of a set of initial conditions. A suitable set of initial conditions is given in Fung (1965). These conditions are

$$\sum_{i=1}^{j} p_i \frac{\mathrm{d}^{i-1}\sigma}{\mathrm{d}t^{i-1}}(0^+) = \sum_{i=1}^{j} q_i \frac{\mathrm{d}^{i-1}\varepsilon}{\mathrm{d}t^{i-1}\varepsilon}(0^+) \quad j = 1, 2, \ldots, m \tag{2}$$

Physically the conditions in eqn (2) mean that there is a direct relation between stress and strain, i.e., there cannot initially exist a strain or a strain derivative without there being any stress or stress derivatives.

A more general constitutive equation of linear viscoelasticity is the convolution integral form

$$\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{-\infty}^{t} E_{\mathrm{rel}}(t-\tau)\varepsilon(\tau)\,\mathrm{d}\tau\right] \tag{3}$$

where $E_{\mathrm{rel}}(t)$ is the stress relaxation funtion (i.e., the stress response on a unit strain deformation). Note that eqn (3) is a more general form than eqn (1) and we will apply eqn (3) on forms that are not reducible to eqn (1). To describe the same viscoelastic behavior as eqn (1) together with the initial conditions in eqn (2) the relaxation modulus should be chosen as

$$E_{\mathrm{rel}}(t) = E - \sum_{k=1}^{N} \Delta E_k (1 - e^{-t/b_k}) \tag{4}$$

Here ΔE_k and b_k are the relaxation strength and the relaxation time corresponding to the k-th dissipative mechanism. The correspondence between eqns (1) and (3) can be shown by applying the Laplace transform to the equations, see Fung (1965). Note that eqn (3) does not need explicit initial conditions since they are automatically accounted for. An alternative form of eqn (3) is obtained by application of the Leibnitz rule

$$\sigma(t) = E\left(\varepsilon(t) - (f * \varepsilon)(t)\right) = E\left(\varepsilon(t) - \int_0^t f(t-\tau)\varepsilon(\tau)\,\mathrm{d}\tau\right) \tag{5}$$

if the initial strain is $\varepsilon(t) = 0$ for $t < 0$. Here $E = E_{\mathrm{rel}}(0)$ is the unrelaxed or instantaneous modulus and $f(t)$ is the kernel or memory function. Causality requirements enforce $f(t)$ to be a causal function (i.e., it vanishes for $t < 0$). The memory function is here related to the relaxation modulus as

$$f(t) = -\frac{1}{E}\frac{\mathrm{d}E_{\mathrm{rel}}(t)}{\mathrm{d}t}, \quad t > 0 \tag{6}$$

3. Frequency domain considerations

The Fourier transform is here defined as

$$\mathscr{F}[x(t)](\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t \tag{7}$$

The correspondence to eqn (5) in the frequency domain, assuming that there exist such a correspondence, can be written as

$$\sigma(\omega) = E(1-F(\omega))\varepsilon(\omega) \equiv E^*(\omega)\varepsilon(\omega) \tag{8}$$

where $\sigma(\omega)$ and $\varepsilon(\omega)$ are the Fourier transforms of stress and strain and $F(\omega)$ is the Fourier transform of the kernel. $E^*(\omega)$ is defined as the complex frequency dependent modulus of elasticity.

If we insist on modeling the viscoelastic behavior in the frequency domain there are a number of basic requirements that must be taken into consideration:

(1) The dissipativity condition states that energy should be removed rather than imposed.
(2) The positivity condition on the relaxed stiffness.
(3) The causality condition which implies that the function $F(\omega)$ must be the Fourier integral of a causal function.

The dissipativity condition and the positivity assumption can be formulated for the complex modulus $E^*(\omega)$ as

$$\Im E^*(\omega) = \Im E(1-F(\omega)) > 0 \quad \text{for}\, \omega > 0 \quad \text{and} \quad \Im E^* \,\text{odd} \tag{9a}$$

$$\Re E^*(0) = \Re E(1-F(0)) > 0 \tag{9b}$$

Here $\Im$ and $\Re$ denote imaginary and real parts, respectively.

We are not aware of any completely general solution to the problem of determining whether $F(\omega)$ is a Fourier transform of a causal function or not. However, if $F(\omega)$ is square integrable ($F(\omega) \in L^2(\mathbb{R})$; for the definition of L^p Banach spaces, see Richtmyer (1978); $\mathbb{R}$ denotes the set of reals), one can establish a necessary and sufficient condition for $F(\omega)$ to be a Fourier integral of a causal function $f(t)$. The real and imaginary part of $F(\omega)$ must then be related to each other as Hilbert transforms and thus satisfy the equations

$$\Im F(\omega) = -\frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty} \frac{\Re F(\Omega)}{\omega-\Omega}\,\mathrm{d}\Omega \tag{10a}$$

$$\Re F(\omega) = \frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty} \frac{\Im F(\Omega)}{\omega-\Omega}\,\mathrm{d}\Omega \tag{10b}$$

where $\mathscr{P}\int$ is a Cauchy principal value. These relations are essentially the Kramers–Kronig relations in electrodynamics.

If $F(\omega) \in L^2(\mathbb{R})$, the Paley–Wiener theorem (see Papoulis, 1962) provides a necessary condition for causality. This theorem states that a necessary and sufficient condition for a real non-negative function $A(\omega) \in L^2(\mathbb{R})$ to be the amplitude of the Fourier transform of a causal function is that

$$\int_{-\infty}^{\infty} \frac{|\ln(A(\omega))|}{\omega^2+1} \mathrm{d}\omega < \infty \tag{11}$$

Taking $A(\omega) = |F(\omega)|$, we have a necessary condition on $F(\omega) \in L^2(\mathbb{R})$ to be the transform of a causal function. One may note two things. The condition on $F(\omega)$ is not a sufficient condition; it can only be used to rule out unsuitable candidates. The reason for this is that the phase function of $F(\omega)$ might be such that $f(t) = \mathscr{F}^{-1}[F(\omega)]$ is non-causal even though $A(\omega) = |F(\omega)|$ satisfies eqn (11). The second point is that the proof of the Paley–Wiener theorem (Papoulis, 1962) involves the construction of a phase function $\theta(\omega)$ which makes $A(\omega)\,\mathrm{e}^{-\mathrm{i}\theta(\omega)}$ the Fourier transform of a causal function. This is useful in a situation where only the amplitude but not the phase of the function $F(\omega)$ is known.

Apart from the Kramers–Kronig relations and the Paley–Wiener theorem there are of course other frequency domain tests available for causality. One such goes back to Jordan's lemma (see Papoulis, 1962). A sufficient condition for the square-integrable function $F(\omega)$ to be the Fourier transform of a causal function is that $F(\omega)$ is analytic in the (closed) lower half plane (for the present definition of the Fourier transform) and tends to zero as $\omega \to \infty$ in that half plane.

4. Time domain considerations

4.1. *Regularity assumptions*

In this section we make some comments on the significance of regularity assumptions imposed on the kernel function $f(t)$. To make a rigorous mathematical analysis of these conditions falls far outside the present study. However, there are some points significant for the choice of damping model that need to be stressed.

Let us from the outset exclude delta function and delta function derivative behavior from the memory kernel; if viscous damping is to be included this can be done simply by adding such a term by hand. The convolution integral we interpret in the ordinary Lebesgue sense.

The first and most important point to recognize is the duality that exists between the regularity of the kernel function on the one hand and the strain history on the other hand. To make the convolution integral well defined for any strain histories lying in, e.g., $L^p(\mathbb{R})$, the memory kernel cannot be chosen arbitrarily. The reason is of course that singularities, or far past history, of the strain can make the integral divergent if care is not taken in the choice of memory kernel. Essentially, the greater the freedom in strain histories, the less freedom in the choice of kernel function. (For strain histories in $L^p(\mathbb{R})$, it is sufficient that $f(t) \in L^q(\mathbb{R})$, with $p^{-1}+q^{-1} = 1+r^{-1}$, for the convolution to be well defined and belong to $L^r(\mathbb{R})$.) To take one extreme example of this, we note that if there is no restriction at all on the size of the strain in the far past (in which case ε need not $\in L^p(\mathbb{R})$ for any p), the memory kernel must be of bounded support, see Högfors and Andersson (1992). This means that in such a case the memory can only extend a finite time into the past.

A second point to be recognized is that if one insists on doing the modeling of the damping mechanism in the frequency domain rather than in the time domain, there are pitfalls that must be avoided, apart from the ones connected to the causality requirement as discussed above. An example of this is the following. Let us say that we for some reason, e.g., because of duality requirements, know that $f \in L^1(\mathbb{R})$. This has certain interesting consequences. By the Riemann–Lebesgue lemma, see Reed and Simon (1975), the Fourier transform $f(t)$ will be in $C_\infty(\mathbb{R})$, the space of continuous functions vanishing at infinity. A discontinuous imaginary part of the complex modulus is in such a case not possible! Furthermore, we may note that the mapping $L^1(\mathbb{R}) \xrightarrow{\mathscr{F}} C_\infty(\mathbb{R})$ is not surjective, i.e., even if we do choose a continuous complex modulus, even one which complies with the requirement of vanishing at infinity, we can still not be sure that the time domain memory kernel belongs to $L^1(\mathbb{R})$. As another example: in Fabrizio et al. (1994), it is argued that minimal assumptions on the memory kernel are that it should lie not merely in $L^1(\mathbb{R})$, but in $L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$. This has certain consequences, one of them being that if $f(0^+)$ exists, the Fourier transform of the memory kernel cannot be chosen arbitrarily, but must be in $L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$. (Under these more restrictive assumptions, it is then straightforward to formulate relations which must hold for the Fourier transform of the kernel, like the Kramers–Kronig relations given above.)

4.2. *Time domain dissipativity*

In this section we give restrictions on the stress relaxation function for the constitutive relation to be dissipative. The internal or dissipative work done in any finite process starting from the material virgin state to time t must be non-negative

$$W = \int_0^t \sigma(s) \frac{\partial \varepsilon(s)}{\partial s} \mathrm{d}s = \int_0^t \int_0^t E_{\mathrm{rel}}(s-\tau) \frac{\partial \varepsilon(\tau)}{\partial \tau} \frac{\partial \varepsilon(s)}{\partial s} \mathrm{d}\tau \, \mathrm{d}s \geqslant 0 \tag{12}$$

where the constitutive relation in eqn (3) has been used. Here it is assumed that the strain is zero for negative times and that E_{rel} is and even function (since it is a causal function). It follows from the Bochner–Schwartz theorem that the dissipation inequality eqn (12) is satisfied if the kernel E_{rel} is a function of positive type (Reed and Simon, 1975). This condition is satisfied, e.g., if (see Rabotnov, 1980; Breuer and Onat, 1962)

- E_{rel} is a non-negative bounded function montonically decreasing as $t \to \infty$
- E_{rel} is convex from below for $t > 0$.

4.3. *Fading memory*

A third point to recognize is that any reasonable memory kernel should of course in some sense represent a fading memory. In the present paper we discuss a few examples of memory kernels corresponding to different damping models. It should be noted that in these examples the fading memory requirement is satisfied in the rather strict sense that the memory kernel $f(t)$ is strictly monotonic and decreasing towards zero as a function of time. Conditions for complete monotonicity of the kernel function is discussed by Day (1967). Fading memory properties of systems and materials are throughout fully discussed by Coleman and Mizel (1967) more recently by Fabrizio et al. (1994).

A survey of restrictions on kernel functions from thermodynamic considerations are given in the book by Day (1972).

5. Exponentially decaying memory kernel

For a material having monotonically exponentially decaying memory the causal kernel takes the form [cf eqns (4) and (6)]

$$f(t) = \sum_{k=1}^{m} \left(\frac{1}{T_k} \mathrm{e}^{-t/b_k} \right), \quad t > 0 \tag{13}$$

which is in $L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$. Here b_k are the relaxation times and T_k are positive constants of dimension time. The time constants may be expressed as

$$T_k = \frac{E b_k}{\Delta E_k} \tag{14}$$

The kernel in eqn (13) can be thermodynamically motivated from irreversible thermodynamics by the introduction of a physically significant internal variable, see, e.g., Nowick and Berry (1972) and Biot (1956).

The stress–strain relation in the frequency domain becomes

$$\sigma(\omega) = E^*(\omega)\varepsilon(\omega) = E\left[1 - \sum_{k=1}^{m} \left(\frac{b_k}{T_k} \left(\frac{1}{1+\omega^2 b_k^2} \right) - \mathrm{i} \frac{b_k}{T_k} \left(\frac{\omega b_k}{1+\omega^2 b_k^2} \right) \right] \varepsilon(\omega) \tag{15}$$

The parameters T_k and b_k are fitted to measured data. Note that to satisfy the positivity assumption on the relaxed stiffness eqn (9b) the parameters must be chosen so that E_∞ is positive

$$E_\infty = E - E \sum_{k=1}^{m} b_k/T_k = E - \sum_{k=1}^{m} \Delta E_k > 0 \tag{16}$$

The causality requirement is naturally fulfilled due to the fact that the original relations in the time domain are causal. For high frequencies $\omega \to \infty$ the resulting modulus $E^*(\omega)$ approaches the instantaneous modulus E and for low frequencies $\omega \to 0$ the modulus approaches the long time or relaxed modulus E_∞.

6. Fractional derivative model of viscoelasticity

To solve the structural equations of motion including a viscoelastic material with a constitutive relation involving several memory kernels is an intractable task. A fractional order derivative model of viscoelasticity is known to require fewer parameters, typically four, to model the actual weak frequency dependence of the complex modulus for engineering materials. Bagley and Torvik (1983) reached a good agreement when they fitted their fractional calculus model to measured data for an elastomer.

The simplest fractional derivative model of viscoelasticity takes the form

$$\sigma(t)+b^{\alpha}\,\mathrm{D}^{\alpha}\sigma(t)=E_{\infty}\varepsilon(t)+Eb^{\alpha}\,\mathrm{D}^{\alpha}\varepsilon(t) \tag{17}$$

where b can be seen as a (generalized) relaxation time with dimension time and D^{α} is a generalized differentiation operator of order α. Note that the relation eqn (17) is not unique. Since derivative operators are involved there is a need for an initial condition. Differentiation of general order α can be defined as the convolution, see Gel'fand and Shilov (1964),

$$\bar{\mathrm{D}}^{\alpha}x(t)\equiv(x*\Phi_{-\alpha})(t),\quad \alpha>0 \tag{18}$$

where

$$\Phi_{-\alpha}(t)=\frac{t_{+}^{-1-\alpha}}{\Gamma(-\alpha)} \tag{19}$$

where Γ is the gamma function

$$\Gamma(x)\equiv\int_{0}^{\infty}\mathrm{e}^{-t}t^{x-1}\,\mathrm{d}t,\quad \Re x>0 \tag{20}$$

with analytical continuation to $\Re[x]\leqslant 0$, $x\notin 0,\ -1,\ -2,\ -3\ldots$ and t_{+} is,

$$t_{+}=\begin{cases}t & t>0\\ 0 & t<0\end{cases} \tag{21}$$

The operator $\bar{\mathrm{D}}^{\alpha}$ defined above is a distributional derivative operator [i.e., it incorporates the behavior at the singularity point ($t=0$) explicitly]. The convolution integral is divergent and must be interpreted in the sense of its regularization. The same definition applies for integration of fractional order by a formal replacement of α by $-\alpha$, the integral expression is then convergent for sufficiently regular $x(t)$. A convergent integral expression for ordinary differentiation of fractional order can be obtained by first applying a derivative of integer order and then integrating of fractional order. The trick is to write the fractional order operator as

$$\mathrm{D}^{\alpha}=\mathrm{D}^{N-\rho}=\mathrm{D}^{N}\,\mathrm{D}^{-\rho} \tag{22}$$

where N is an integer that satisfies $\alpha<N\leqslant\alpha+1$ and $0<\rho\leqslant 1$. The last factor in eqn (22) is in fact an fractional integration. Differentiation of arbitrary order can now be defined by a convergent integral expression as

$$\mathrm{D}^{\alpha}x(t)\equiv\mathrm{D}^{N-\rho}x(t))\equiv\frac{\mathrm{d}^{N}}{\mathrm{d}t^{N}}\left[\frac{1}{\Gamma(\rho)}\int_{0}^{t}(t-\tau)^{(\rho-1)}x(\tau)\,\mathrm{d}\tau\right],\quad 0<\rho\leqslant 1 \tag{23}$$

The trick above, eqn (22), can be justified by the fact that $\mathrm{D}^{N-\rho}$ can be taken as the analytical continuation of D^{q}, see Ross (1975). For a causal function the distributional fractional order derivative is equal to the ordinary fractional order derivative with the exception of points where the ordinary fractional derivative does not exist.

Specializing to $0<\alpha<1$ which is the interesting interval for the application of fractional calculus to viscoelasticity, the definition of fractional derivative can be written as

$$\mathrm{D}^{\alpha}x(t) \equiv \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}t\right], \quad 0<\alpha<1 \tag{24}$$

One may note that the fractional differential operator is not a local operator, i.e., the derivative is not only dependent of the value at the point but the value of the function on the whole interval. As seen from the definition in eqn (23) the fractional differentiation operator produces the same results as the ordinary operator of integer order if α is a positive integer. This is true also for the definition in eqn (18), Gel'fand and Shilov (1964).

6.1. *Frequency and Laplace domain properties*

In the time domain the presence of integro-differential operators makes the solution of the structural equations computationally more complex than do the ordinary operators. However, in the frequency and the Laplace domains the model becomes easy to handle, apart from the question how to choose the branch of the root in the Fourier transform of the fractional derivative operator.

The Laplace transform is here defined as

$$\mathscr{L}[x(t)](s) = \int_{0^+}^{\infty} x(t)\,\mathrm{e}^{st}\,\mathrm{d}t \tag{25}$$

When $x(t)$ or its derivatives have singularities at $t=0$, it is necessary to choose between taking either 0^+ or 0^- as lower limit in the integral in eqn (25). The Laplace transform of the (ordinary) fractional order derivative is, see Oldham and Spanier (1974),

$$\mathscr{L}[\mathrm{D}^{\alpha}x(t)] = s^{\alpha}\mathscr{L}[x(t)] - \sum_{k=0}^{n-1} s^{k}\,\mathrm{D}^{\alpha-1-k}x(0^+) \tag{26}$$

where n is an integer such that $n-1<\alpha\leqslant n$. Note the occurrence of initial conditions of fractional order in eqn (26). One might argue that with a translation of the time scale one can always prescribe homogeneous initial conditions i.e., $\mathrm{D}^{\alpha-1-k}x(0^+)=0$ for $k\in[0,n-1]$. However, by using theorems for translation and transformation for the derivative of general order we have

$$\mathscr{L}[{}_0\mathrm{D}^{\beta}H(t-t_0)x(t)] = s^{\beta}\mathscr{L}[H(t-t_0)x(t)] - \mathrm{e}^{-st_0}\sum_{k=0}^{n-1} s_{t_0}^{k}\,\mathrm{D}^{\beta-1-k}x(t_0^+) \tag{27}$$

where $n-1<\beta\leqslant n$. Index in lower left of derivative operator denotes lower limit in the convolution integral defining the fractional order differentiation operator see eqn (23). It is thus not possible to overcome the problem of the initial values by a translation of the time scale. The Laplace transformation of the distributional derivative operator is simply

$$\mathscr{L}[\bar{\mathrm{D}}^{\alpha}x(t)] = s^{\alpha}\mathscr{L}[x(t)] \tag{28}$$

A unique expression for the Fourier transformation of the fractional derivative operator is, see Gel'fand and Shilov (1964), (note the difference in the definitions of the Fourier transform between Gel'fand and Shilov and the present paper)

$$\mathscr{F}[\bar{\mathrm{D}}^{\alpha}x(t)] = \mathrm{e}^{i\alpha\pi/2}(\omega - \mathrm{i}0^{+})^{\alpha}\mathscr{F}[x(t)] \tag{29}$$

where $(\omega - \mathrm{i}0^{+})^{\alpha}$ should be interpreted as

$$(\omega - \mathrm{i}0^{+})^{\alpha} = \begin{cases} \mathrm{e}^{-\mathrm{i}\alpha\pi}|\omega|^{\alpha} & \omega < 0 \\ \omega^{\alpha} & \omega > 0 \end{cases} \tag{30}$$

From now on we write the Fourier transformation of the fractional derivative of order α

$$\mathscr{F}[\bar{\mathrm{D}}^{\alpha}x(t)] = (\mathrm{i}\omega)^{\alpha}\mathscr{F}[x(t)] \tag{31}$$

with the understanding that $(\mathrm{i}\omega)^{\alpha}$ should be interpreted as $\exp(\mathrm{i}\alpha\pi/2)(\omega - \mathrm{i}0^{+})^{\alpha}$.

The frequency domain constitutive equation corresponding to the fractional derivative model of viscoelasticity can now be written as

$$\sigma(\omega) = E\left(1 - \frac{1 - E_{\infty}/E}{1 + (b\mathrm{i}\omega)^{\alpha}}\right)\varepsilon(\omega) = E(1 - F(\omega))\varepsilon(\omega) \tag{32}$$

Figure 1 shows the frequency dependence of the loss factor corresponding to the complex modulus of the fractional derivative model in eqn (17) for different values of $\alpha \in (0, 1]$. A small value for α gives a weak frequency dependence of the loss factor.

6.2. *Fractional derivative model on convolution integral form*

The Laplace domain constitutive equation corresponding to the fractional derivative model of viscoelasticity, eqn (17), can be written as

$$\sigma(s) = E\left(1 - \frac{1 - E_{\infty}/E}{1 + (bs)^{\alpha}}\right)\varepsilon(s) = E(1 - F(s))\varepsilon(s) \tag{33}$$

together with the initial condition

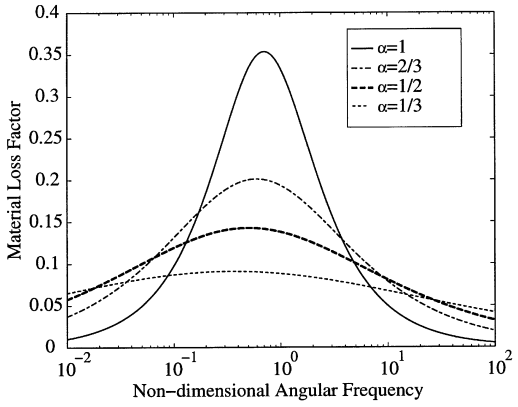

Fig. 1. Loss factor vs normalized angular frequency ωb. The influence of different values of $\alpha \in (0, 1]$, is shown for $E_{\infty}/E = 1/2$.

$$\mathrm{D}^{-(1-\alpha)}\sigma(0^+) = E\mathrm{D}^{-(1-\alpha)}\varepsilon(0^+) \tag{34}$$

which, for $0 < \alpha < 1$, is a fractional integrated form of Hooke's law and thus a weaker condition than Hooke's law.

The correspondence to eqn (17) on hereditary or integral form can be formally obtained from eqn (33) by use of the Laplace convolution theorem, assuming that the convolution is well defined,

$$\sigma(t) = E(\varepsilon(t) - (f_\alpha * \varepsilon)(t)) \quad t > 0 \tag{35}$$

where $f_\alpha(t)$ is the inverse Laplace transform of $F(s)$ in eqn (33). $F(s)$ is not in $L^1(\mathbb{R})$ but it is in $L^p(\mathbb{R})$ for $p > 1/\alpha$. The inverse can be found as

$$f_\alpha(t) = \mathscr{L}^{-1}\left[\frac{1 - E_\infty/E}{1 + (bs)^\alpha}\right](t), \quad t > 0 \tag{36}$$

Take $c > 0$ so that $|(bc)^{-\alpha}| < 1$. Along the vertical line $c - \mathrm{i}\infty$ to $c + \mathrm{i}\infty$ we write:

$$\frac{1}{1 + (bs)^\alpha} = \frac{1}{(bs)^\alpha}\left(\frac{1}{1 + (bs)^{-\alpha}}\right) = \sum_{n=0}^{\infty} (-1)^n ((bs)^\alpha)^{-n-1} \tag{37}$$

which converges uniformly on the vertical line in consideration. For $t > 0$ $f_\alpha(t)$ can then be written as a line integral:

$$\begin{aligned} f_\alpha(t) &= \mathscr{L}^{-1}[F_\alpha(s)](t) = \frac{1}{2\pi i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F_\alpha(s)\,\mathrm{e}^{st}\,\mathrm{d}s \\ &= \left(1 - \frac{E_\infty}{E}\right)\sum_{n=0}^{\infty}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}(-1)^n((bs)^\alpha)^{-n-1}\,\mathrm{e}^{st}\,\mathrm{d}s \\ &= \left(1 - \frac{E_\infty}{E}\right)\sum_{n=0}^{\infty}\mathscr{L}^{-1}[(-1)^n((bs)^\alpha)^{-n-1}], \quad t > 0 \end{aligned} \tag{38}$$

By use of a standard table of Laplace transforms, see Oberhettinger and Badii (1973) p. 237, we obtain

$$\begin{aligned} f_\alpha(t) &= \left(1 - \frac{E_\infty}{E}\right)\sum_{n=0}^{\infty}(-1)^n b^{-1}\frac{(t/b)^{\alpha(n+1)-1}}{\Gamma(\alpha(n+1))} \\ &= \left(1 - \frac{E_\infty}{E}\right)\sum_{n=1}^{\infty}(-1)^{n+1} b^{-1}\frac{(t/b)^{\alpha n-1}}{\Gamma(\alpha n)}, \quad t > 0 \end{aligned} \tag{39}$$

or, in Mittag–Leffler notation

$$f_\alpha(t) = \left(1 - \frac{E_\infty}{E}\right)\left(-\frac{\mathrm{d}}{\mathrm{d}t}[E_\alpha(-(t/b)^\alpha)]\right) \tag{40}$$

where E_α is the α-order Mittag–Leffler function, which is defined as (Bateman, 1955)

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1+\alpha n)} \tag{41}$$

Kernel functions of Mittag–Leffler function type were introduced into linear viscoelasticity by Rabotnov (1980) and the connection to the use of fractional derivative operators in constitutive equations of linear viscoelasticity was established by Koeller (1984).

The asymptotic behavior of $f_\alpha(t)$ as $n \to \infty$ is obtained by use of Stirling's formula: (Abramowitz and Stegun, 1965)

$$\Gamma(z) \sim \sqrt{2\pi}\, \mathrm{e}^{-z} z^{z-1/2}, \quad |z| \to \infty \quad \text{and } |\arg| < \pi \tag{42}$$

The asymptotic behavior of the nth term is then

$$\begin{aligned} f_\alpha^n(t) &\equiv \left(1 - \frac{E_\infty}{E}\right)(-1)^{n+1} b^{-1} \frac{(t/b)^{\alpha n - 1}}{\Gamma(\alpha n)} \\ &\sim \left(1 - \frac{E_\infty}{E}\right)(-1)^n \frac{b^{-1}(t/b)^{\alpha n - 1}}{\sqrt{2\pi}\, \mathrm{e}^{-n\alpha}(n\alpha)^{n\alpha - 1/2}} \quad \text{as } n \to \infty \end{aligned} \tag{43}$$

For a given t the ratio test gives

$$\left|\frac{f_\alpha^{n+1}}{f_\alpha^n}\right| \sim \left(\frac{t/b}{(n+1)\alpha}\right)^\alpha \to 0 \quad \text{as } n \to \infty \tag{44}$$

so that the series is convergent. But, as seen from the ratio above, the convergence is very poor. For a given t we need to add

$$n > \frac{(t/b)}{\alpha} - 1 \tag{45}$$

terms before we can expect the terms to begin to fall off in size. If a high number of term is to be included in the sum when evaluating $f_\alpha(t)$ the numerical stability is lost.

Consider the case of $\alpha = 1$, then eqn (39) becomes

$$f_1(t) = \left(1 - \frac{E_\infty}{E}\right)\frac{1}{b}\sum_{n=1}^{\infty}\left((-1)^{n+1}\frac{(t/b)^{(n-1)}}{\Gamma(n)}\right) = \left(1 - \frac{E_\infty}{E}\right)\frac{1}{b}\mathrm{e}^{-t/b}, \quad t > 0 \tag{46}$$

which is the exponentially decaying memory kernel as expected. Specializing eqn (39) to $\alpha = 1/2$ gives for $t > 0$

$$f_{1/2}(t) = \left(1 - \frac{E_\infty}{E}\right)\frac{1}{b}\left(\frac{1}{\sqrt{\pi t/b}} - \mathrm{e}^{t/b}\,\mathrm{erfc}(\sqrt{t/b})\right) \tag{47}$$

where $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ is the complementary error function.

Due to the "symmetry" between stress and strain in the formulation of the fractional derivative constitutive equation of viscoelasticity eqn (17) we can easily formulate the inverse to eqn (35),

$$\varepsilon(t) = \frac{1}{E}(\sigma(t) + (g_\alpha * \sigma)(t)), \quad t > 0 \tag{48a}$$

$$g_\alpha(t) = \frac{1}{l}\left(\frac{E}{E_\infty} - 1\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(t/l)^{\alpha n-1}}{\Gamma(\alpha n)}, \quad t > 0 \tag{48b}$$

where $l = b(E/E_\infty)^{1/\alpha}$ is the "generalized" relaxation time with respect to constant stress.

6.3. *Asymptotic expansion of the memory kernel*

From reasons mentioned in Section (6.2) it is obvious that the expression for $f_\alpha(t)$, eqn (39), is not of practical use for increasing values of t, and there is a need for an asymptotic expression for $f_\alpha(t)$ for large times. Concerning the asymptotic behavior of the functions $f_\alpha(t)$, we utilize a formal mode of reasoning which in van der Pol and Bremmer (1975) is attributed to Heaviside. (For a recent application of this mode of reasoning to wave splitting in structural dynamics, see Olsson and Kristensson, 1994).

Expanding the Laplace transform of $f_\alpha(t)$ according to eqn (39) for small $s > 0$ gives

$$\mathscr{L}[f_\alpha(t)](s) = \left(1 - \frac{E_\infty}{E}\right)\sum_{n=0}^{\infty}(-1)^n(bs)^{\alpha n} \tag{49}$$

with convergence in a radially cut circular region of radius b. Formal term-wise inversion then yields an asymptotic series for $f_\alpha(t)$ as $t \to \infty$:

$$f_\alpha(t) \sim \left(1 - \frac{E_\infty}{E}\right)\sum_{n=1}^{\infty}(-1)^n b^{-1}\frac{(t/b)^{-\alpha n-1}}{\Gamma(-\alpha n)} \quad \text{as } t \to \infty \tag{50}$$

where the term $n = 0$, and in fact any term with integer αn, drops out of the asymptotic series due to the fact that $t > 0$ is considered. (They represent terms with support concentrated at $t = 0$.)

We emphasize that the above derivation is purely formal and requires detailed verification which we will not attempt here, see van der Pol and Bremmer (1975). However, numerical calculations verify that the sum of the first few terms in the formal asymptotic series do indeed approximate $f_\alpha(t)$ well for large t. Further, for the case of $\alpha = 1/2$ the asymptotic series according to eqn (50) correctly reproduces the result of inserting the asymptotic expansion of $e^t \operatorname{erfc}(\sqrt{t})$, see Abramowitz and Stegun (1965), into eqn (47):

$$f_{1/2}(t) \sim -\left(1 - \frac{E_\infty}{E}\right)\frac{1}{b}\frac{1}{\sqrt{\pi t/b}}\sum_{m=1}^{\infty}(-1)^m\frac{(2m-1)!!}{(2t/b)^m} \quad \text{as } t \to \infty \tag{51}$$

Note that the asymptotic expansion can speed up the calculations of $f_\alpha(t)$ for "large" times considerably, as well as improve the numerical stability of the calculation. Instead of having to sum perhaps hundreds of terms in the "one-sided Taylor expansion" of f_α in eqn (39), just a few terms of the asymptotic series may suffice.

6.4. *Regularity properties of the memory kernel*

To what Banach space (or spaces) $L^p(\mathbb{R})$ does $f_\alpha(t)$ belong? It is even integrable? According to eqn (39) and the asymptotic series in eqn (50), we have

$$f_\alpha(t) \sim \begin{cases} \mathcal{O}(t^{\alpha-1}), & t \to 0^+ \\ \mathcal{O}(t^{-\alpha-1}), & t \to +\infty \end{cases} \tag{52}$$

For $\alpha = 1$ the behavior at temporal infinity is even better, but as can easily be seen, it is the short time behavior which is crucial here. Since $f_\alpha(t)$ is also continuous between these limits (and vanishes for $t < 0$), every $f_\alpha(t)$ belongs to $L^1(\mathbb{R})$. The continuity follows from the fact that $tf_\alpha(t)$ for $t > 0$ coincides with an analytical function in the variable $z = t^\alpha$ with a Laurent series convergent in the annular region $0 < |z| < \infty$. However, from the above considerations we also find that if $0 < \alpha < 1$

$$f_\alpha(t) \in L^p(\mathbb{R}) \quad \text{for } 1 \leqslant p < \frac{1}{1-\alpha} \tag{53}$$

while

$$f_\alpha(t) \notin L^p(\mathbb{R}) \quad \text{for } p \geqslant \frac{1}{1-\alpha} \tag{54}$$

For $\alpha = 1$, $f_\alpha(t)$ lies in all of the $L^p(\mathbb{R})$ spaces, including $L^\infty(\mathbb{R})$. The dual restrictions on the strain histories can now be formulated as: for a given $\alpha \in (0, 1)$, the strain histories should at least belong to $L^q_{\text{loc}}(\mathbb{R})$ for some $q > 1/\alpha$ and should of course vanish for $t < 0$. This is to ensure that the integrand in the convolution integral in eqn (35), $(f_\alpha * \varepsilon)$, is at least in L^1_{loc}. Note that, since no restriction has been put on the long time behavior of the stress, there is no need to demand that the integrand should be in $L^1(\mathbb{R})$.

Equations (53) and (54) have among their consequences that

$$f_\alpha(t) \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R}) \quad \text{only for } \alpha > \tfrac{1}{2} \tag{55}$$

i.e., that only for fractional order of derivative strictly greater than 1/2 does the kernel satisfy what in Fabrizio et al. (1994) are designated as minimal assumptions on a memory kernel. Note that for $\alpha \leqslant 1/2$ the Kramers–Kronig relations eqns (10a) and (10b) do not hold between the real and imaginary parts of the Fourier transform of $f_\alpha(t)$ (at least not in the ordinary sense), for the reason that neither of them lies in $L^2(\mathbb{R})$ and are not in the usual domain of definition of the Hilbert transform.

6.5. *Fractional derivatives of the memory kernel*

In Oldham and Spanier (1974) it is noted that certain functions, which with proper choice of a parameter are proportional to $f_\alpha(t)$, are eigenfunctions of the operator D^α. It is worthwhile to elaborate a little on this property of the $f_\alpha(t)$, as it is connected to some possibly useful formulae for the time evolution of the convolution term in the constitutive relation. Let us first consider the convolution term $f_\alpha * \varepsilon(t)$. Its fractional derivative of order α satisfies the following simple relation:

$$b^{\alpha}\,\mathrm{D}^{\alpha}(f_{\alpha}*\varepsilon)(t) = -(f_{\alpha}*\varepsilon)(t)+\left(1-\frac{E_{\infty}}{E}\right)\varepsilon(t) \tag{56}$$

This relation is connected to the fact that the constitutive equation on integral form, eqn (35), with the memory kernel $f_{\alpha}(t)$ according to eqn (39) when substituted into the original fractional derivative constitutive equation, eqn (17), should reduce it to an identity [under the initial condition eqn (34)]. The equation above can readily be demonstrated by Laplace transform methods, or alternatively by noting that

$$(f_{\alpha}*\varepsilon)(t) = \left(1-\frac{E_{\infty}}{E}\right)\sum_{n=1}^{\infty}(-1)^{n+1}b^{-n\alpha}\,\mathrm{D}^{-n\alpha}\varepsilon(t) \tag{57}$$

which follows immediately from the series expansion of $f_{\alpha}(t)$ together with the observation that for negative orders, and sufficiently well behaved $\varepsilon(t)$, the distributional derivative (which is then a fractional integration) coincides with the ordinary derivative since no regularization of the defining convolution integral is required in that case. Operating with D^{α} on this, term-wise evaluation requires evaluation of the composition $\mathrm{D}^{\alpha}\mathrm{D}^{-n\alpha}$. In the general case the composition rule for fractional derivatives is a bit complicated, see Oldham and Spanier (1974). However, in the case which is of interest here, $-n\alpha < 0$ which allows us to infer that $\mathrm{D}^{\alpha}\mathrm{D}^{-n\alpha}(t) = \mathrm{D}^{-(n-1)\alpha}\varepsilon(t)$, which immediately yields eqn (56).

If $f_{\alpha}(t)$ is recognized as an eigenfunction of D^{α}, it may seem strange how eqn (56) could possibly hold; the problem is the last term on the right hand side of the expression. Note however that $f_{\alpha}(t)$ satisfies not only the equation

$$\mathrm{D}^{\alpha}f_{\alpha}(t) = -\frac{1}{b^{\alpha}}f_{\alpha}(t), \quad t>0 \tag{58}$$

but also the initial condition

$$\mathrm{D}^{-(1-\alpha)}f_{\alpha}(0^{+}) = \frac{1}{b^{\alpha}}\left(1-\frac{E_{\infty}}{E}\right) \tag{59}$$

in addition to being causal. In terms of the distributional derivative this can be summed up in the relation

$$b^{\alpha}\,\bar{\mathrm{D}}^{\alpha}f_{\alpha}(t) = -f_{\alpha}(t)+\left(1-\frac{E_{\infty}}{E}\right)\delta(t) \tag{60}$$

which is consistent with eqn (56).

Another way of arriving at eqn (60) is by applying $\bar{\mathrm{D}}^{\alpha}$ to the series expansion of $f_{\alpha}(t)$ and making use of the fact, see Gel'fand and Shilov (1964),

$$\bar{\mathrm{D}}^{\alpha}\left[\frac{t_{+}^{n\alpha-1}}{\Gamma(n\alpha)}\right] = \begin{cases}\dfrac{t_{+}^{(n-1)\alpha-1}}{\Gamma((n-1)\alpha)} & n\geqslant 2\\ \delta(t) & n=1\end{cases} \tag{61}$$

It is of course the $n = 1$ term which produces the delta term in eqn (60). But there is in fact no contradiction between eqn (60) and the fact that $f_\alpha(t)$ is an eigenfunction of D^α. Note that on the interval $t > 0$ eqn (60) indeed states that $\mathrm{D}^\alpha f_\alpha(t)$ is proportional to $f_\alpha(t)$. The presence of the delta distribution term in the equation simply reflects the fact that the fractional derivative operator in Gel'fand and Shilov (1964) is a distributional derivative operator, and thus incorporates the behavior at $t = 0$ explicitly. On the interval $t > 0$, $f_\alpha(t)$ is thus indeed an eigenfunction of the fractional derivative operator of order α, but as far as evaluation of convolutions with $f_\alpha(t)$ is considered, the delta distribution term is essential, as seen in eqn (56).

Consider the causal Green's function corresponding to the operator $\bar{\mathrm{D}}^\alpha + b^{-\alpha}$. The Green's function is the solution to

$$\begin{aligned} [\bar{\mathrm{D}}^\alpha + b^{-\alpha}]g(t,t') &= \delta(t-t') \\ g(t,t') &= 0, \quad t < t' \end{aligned} \tag{62}$$

For $t' = 0$ this can be compared to eqn (60), which indicates that

$$f_\alpha(t) = \left(1 - \frac{E_\infty}{E}\right)\frac{1}{b^\alpha} g(t, 0^+) \tag{63}$$

However, the composition of a translation and the fractional derivative is in general a non-trivial operation, see Gel'fand and Shilov (1964). The reason why this is a non-trivial composition is the lower limit of integration in the definition of fractional derivative. Gel'fand and Shilov (1964) gives a rather forbidding expression for the fractional derivative of a function with translated argument, which for lower limit zero in the definition of the fractional derivative reads:

$$\mathrm{D}^\alpha f(t+A) = \frac{\mathrm{d}^\alpha f(t+A)}{[\mathrm{d}(t+A)]^\alpha} - \sum_{k=1}^{\infty} \frac{\mathrm{d}^{\alpha+k} 1}{[\mathrm{d}(t+A)]^{\alpha+k}} \frac{\mathrm{d}^{-k} f(A)}{[\mathrm{d}A]^{-k}} \tag{64}$$

This makes the fractional derivative operator in general translationally non-invariant, and seems to prevent us from immediately inferring that

$$f_\alpha(t-t') = \left(1 - \frac{E_\infty}{E}\right)\frac{1}{b^\alpha} g(t, t') \tag{65}$$

But nevertheless, eqn (65) is in fact correct. To see that $f_\alpha(t-t')$ (apart from normalization) is the causal Green's function, we may utilize the fact that, see eqn (18),

$$\bar{\mathrm{D}}^{-n\alpha}\delta(t-t') = (\delta(\hat{t}-t') * \Phi_{-\alpha}(\hat{t}))(t) = \frac{(t-t')_+^{n\alpha-1}}{\Gamma(n\alpha)} \tag{66}$$

to rewrite eqn (39) as

$$f_\alpha(t-t') = \left(1 - \frac{E_\infty}{E}\right) \sum_{n=1}^{\infty} (-1)^{n+1} b^{-\alpha n} \mathrm{D}^{-n\alpha}\delta(t-t') \tag{67}$$

Applying $\bar{\mathrm{D}}^\alpha$ to this equation and composing operators as in the derivation of eqn (56) immediately

yields eqn (65). Thus we see that the translated memory kernel is essentially the causal Green's function to the operator $\bar{\mathrm{D}}^{\alpha}+b^{-1}$.

How can it be that, while it is not in general possible to commute translation and fractional differentiation, it is nonetheless possible to do so in the case of Green's function corresponding to $\bar{\mathrm{D}}^{\alpha}+b^{-\alpha}$? There is a very simple answer to that question: for a causal function (which is a function defined on the real line but vanishing on the negative half axis), fractional differentiation (with lower limit 0, as used in this context) commutes with any translation that does not shift any part of the support of the function into the negative half axis. In particular, translation of a causal function to the right always commutes with $\bar{\mathrm{D}}^{\alpha}$. To see this we can make the following formal calculation for a causal function $h(t)$:

$$\bar{\mathrm{D}}^{\alpha}h(t-t') = \int_0^t h(t-\hat{t}-t')\Phi_{-\alpha}(\hat{t})\,\mathrm{d}\hat{t} = \begin{cases} \displaystyle\int_0^{t-t'} h((t-t')-\hat{t})\Phi_{-\alpha}(\hat{t})\,\mathrm{d}\hat{t} & t \geqslant t' \\ 0 & t < t' \end{cases} \tag{68}$$

where the integrals are assumed to be regularized, and we have assumed $t' > 0$ (i.e., translation to the right).

Apart from its theoretical interest in connecting two different formulations of the fractional derivative model of viscoelasticity, we believe that eqn (56) derived above is of some practical importance in connection with the solution of the structural equations for a transient loaded system involving this kind of viscoelastic material. This we pursue further in Section 9.

6.6. *Stress relaxation function*

Consider a specimen of a viscoelastic material having a constitutive relation involving fractional derivatives as eqn (17). The stress relaxation function is the stress response of the specimen on a unit step strain deformation,

$$\varepsilon(t) = H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{69}$$

The stress relaxation function $\sigma_{\alpha}(t)$ or $(E_{\mathrm{rel}}(t))$ can be obtained by introducing the step deformation in the constitutive relation on integral form in eqn (35), hence

$$\sigma_{\alpha}(t) = E\left(1-\int_0^t f_{\alpha}(\hat{t})\,\mathrm{d}\hat{t}\right), \quad t > 0 \tag{70}$$

Insertion of the memory kernel $f_{\alpha}(t)$ according to eqn (39) and changing order of summation and integration yield ($t > 0$),

$$\sigma_{\alpha}(t) = E\left(1-\left(1-\frac{E_{\infty}}{E}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(t/b)^{\alpha n}}{\Gamma(\alpha n+1)}\right) \tag{71}$$

or in Mittag–Leffler notation

$$\sigma_\alpha(t) = E\left(1-\left(1-\frac{E_\infty}{E}\right)(1-E_\alpha[-(t/b)^\alpha])\right) \tag{72}$$

By use of the same method as in Section (6.2), an asymptotic series for $\sigma_\alpha(t)$ as $t \to +\infty$ is obtained as

$$\sigma_\alpha(t) \sim E\left(1-\left(1-\frac{E_\infty}{E}\right)\left(1+\sum_{n=1}^{\infty}(-1)^n\frac{(t/b)^{-n\alpha}}{\Gamma(-n\alpha+1)}\right)\right) \tag{73}$$

where any term with $\alpha n-1$ equal to an integer drops out of the sum.

The short time ($t \to 0^+$) and long time ($t \to +\infty$) values of the stress relaxation function are obtained from the expressions for $\sigma_\alpha(t)$ and the asymptotic series for $\sigma_\alpha(t)$ as $t \to \infty$ as,

$$\lim_{t\to 0}\sigma_\alpha(t) = E \tag{74a}$$

$$\lim_{t\to \infty}\sigma_\alpha(t) = E_\infty \tag{74b}$$

as of course they should be.

The stress relaxation function eqn (72) satisfies both the time domain dissipativity condition eqn (12) and the fading memory hypothesis in Section 4.3 if the parameters are subjected to the following restrictions $E > E_\infty > 0$, $b > 0$, and $0 < \alpha \leqslant 1$. The reason for this is that the Mittag–Leffler function is a completely monotonic function (for $0 < \alpha \leqslant 1$), (i.e., Bateman, 1955)

$$(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}(E_\alpha[x]) \geqslant 0, \quad n = 1,2,3,\ldots \tag{75}$$

If specializing to $\alpha = 1/2$, $\sigma_{1/2}$ can be expressed in the complementary error function, (erfc) as

$$\sigma_{1/2}(t) = E\left(1-\left(1-\frac{E_\infty}{E}\right)(1-\mathrm{e}^{t/b}\,\mathrm{erfc}(\sqrt{t/b}))\right), \quad t > 0 \tag{76}$$

and if specializing to $\alpha = 1$ we obtain

$$\sigma_1(t) = E\left(1-\left(1-\frac{E_\infty}{E}\right)(1-\mathrm{e}^{-t/b})\right), \quad t > 0 \tag{77}$$

The expression for the stress relaxation function $\sigma_1(t)$ is, as expected, the same as the well-known stress relaxation function corresponding to the Standard Linear Solid.

Figure 2 shows the stress relaxation functions for different $\alpha \in (0, 1]$. Note that the "time" for approaching the asymptotic value of the stress relaxation function as $t \to \infty$ is strongly dependent on which fraction order of differentiation that is implied in the constitutive behavior.

The creep function (i.e. the strain response due a unit stress applied at $t = 0$), is obtained in the same way as stress relaxation function by use of constitutive relation for $\varepsilon(t)$ on integral form eqns (48a) and (48b),

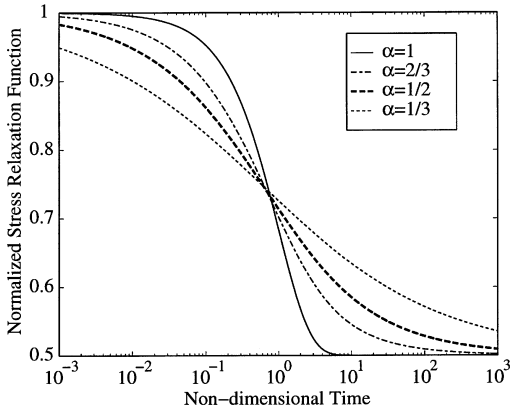

Fig. 2. Normalized stress relaxation functions $\sigma_\alpha(t)/E$ vs non-dimensional time t/b. The influence of different values of $\alpha \in (0, 1]$ is shown for $E_\infty/E = 1/2$.

$$\varepsilon_\alpha(t) = \frac{1}{E}\left(1 + \left(\frac{E - E_\infty}{E_\infty}\right)(1 - E_\alpha[-(t/l)^\alpha])\right), \quad l = b\left(\frac{E}{E_\infty}\right)^{1/\alpha} \tag{78}$$

Figure 3 shows the creep function for different $\alpha \in (0, 1]$.

7. Augmenting thermodynamic fields

Instead of formulating the constitutive equation on integral form or on differential operator form involving derivatives on both stress and strain, the constitutive equation can be formulated as coupled equations by introducing internal variables that can be motivated by irreversible thermodynamics, see Biot (1956) and Nowick and Berry (1972). This model has more recently

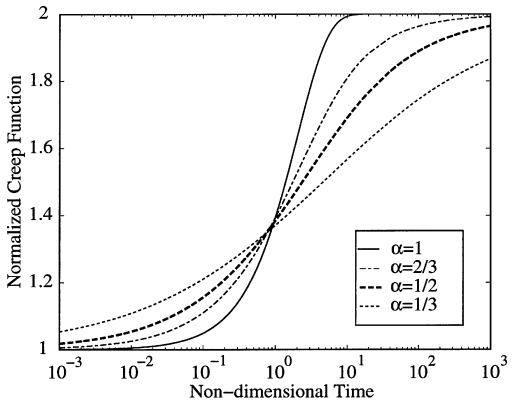

Fig. 3. Normalized creep function $\varepsilon_\alpha/\varepsilon_0$ vs non-dimensional time t/b. Here $\varepsilon_0 = \sigma_0/E$ and σ_0 is the applied stress step. The influence of different values of $\alpha \in (0, 1]$ is shown for $E_\infty/E = 1/2$.

been used by Lesieutre and Mingori (1990) and Leisuetre (1992) to model frequency-dependent material damping in finite element calculations.

The one-dimensional constitutive equation with one internal variable $\xi(t)$ can be written as two coupled equations, see Lesieutre and Mingori (1990),

$$\sigma(t) = E\varepsilon(t) - \delta\xi(t) \tag{79a}$$

$$A(t) = \delta\varepsilon(t) - \gamma\xi(t) \tag{79b}$$

where $A(t)$ is the field thermodynamically conjugate to $\xi(t)$, δ is the material constant that relates the two coupled variables $\varepsilon(t)$ and $\xi(t)$, and γ is a material constant. A reasonable assumption is that there is a definite equilibrium value of $\xi(t)$ denoted $\bar{\xi}(t)$. $\bar{\xi}(t)$ is the value of $\xi(t)$ when $A(t) = 0$, hence

$$\bar{\xi} = \frac{\delta}{\gamma}\varepsilon(t) \tag{80}$$

Assume that $\mathrm{d}\xi(t)/\mathrm{d}t$ is proportional to the negative of its deviation from its equilibrium value $\bar{\xi}(t)$. The governing equation for the thermodynamical field can then be written as

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = -\frac{1}{b}(\xi(t) - \bar{\xi}(t)) = -\frac{1}{b}\left(\xi(t) - \frac{\delta}{\gamma}\varepsilon(t)\right) \tag{81}$$

where b is relaxation time. Since eqn (81) is a differential equation it naturally needs an initial condition. The following initial condition is consistent with an instantaneous response following Hooke's law [which is seen by simply introducing the initial condition into eqn (79a)]

$$\xi(0) = 0 \tag{82}$$

Equations (79a) and (81) are sufficient to describe the viscoelastic behavior. By eliminating $\xi(t)$ from eqns (79a) and (81) we obtain

$$\sigma(t) + b\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = \left(E - \frac{\delta^2}{\gamma}\right)\varepsilon(t) + Eb\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} \tag{83}$$

which is the constitutive equation for the Standard Linear Solid on differential operator form with relaxed modulus $E_\infty = E - \delta^2/\gamma$.

For materials with weaker frequency dependence of its dynamic properties than the Standard Linear Solid, several internal fields ξ_i representing different internal mechanisms can be introduced. The governing viscoelastic equations then take the form

$$\sigma(t) = E\varepsilon(t) - \sum_{i=1}^{N} \delta_i\xi_i(t) \tag{84a}$$

$$\frac{\mathrm{d}\xi_i(t)}{\mathrm{d}t} = \frac{1}{b_i}\left(\xi_i(t) - \frac{\delta_i}{\gamma_i}\varepsilon(t)\right), \quad i = 1, \ldots, N \tag{84b}$$

This form is equivalent to several exponentially decaying memory kernels in the convolution integral formulation of the constitutive equation.

Instead of involving several internal variables one might try to include a whole spectrum of effects in one field. The fractional order derivative of this field is taken to be proportional to the negative of its deviation from its equilibrium value,

$$\mathrm{D}^{\alpha}\xi(t) = -\frac{1}{b^{\alpha}}\left(\xi(t) - \frac{\delta}{\gamma}\varepsilon(t)\right) \tag{85}$$

The appropriate initial condition to eqn (85) is

$$\mathrm{D}^{-(1-\alpha)}\xi(0^{+}) = 0 \tag{86}$$

which may be seen by applying the Laplace transform to eqn (85). By eliminating $\xi(t)$ from eqns (79a) and (85) we obtain

$$\sigma(t) + b\,\mathrm{D}^{\alpha}\sigma(t) = \left(E - \frac{\delta^2}{\gamma}\right)\varepsilon(t) + Eb\,\mathrm{D}^{\alpha}\varepsilon(t) \tag{87}$$

which is the same constitutive equation as the fractional derivative constitutive equation on differential operator form eqn (17). The coupled equations (79a) and (85) are thus an alternative form to describe the same constitutive model as the fractional derivative constitutive equation of viscoelasticity, eqn (17), or the convolution integral constitutive equation, eqn (35), with a memory kernel according to eqn (39).

8. Structural equations of motion

In this section we will discuss and give some remarks on the formulation of the structural equations of motions when the fractional order derivative model of viscoelasticity is employed for modeling the damping. Consider a one-degree of freedom system consisting of a discrete mass and a viscoelastic spring having a fractional derivative constitutive law. The dynamic equation for the mass:

$$m\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2} + r(t) = R(t) \tag{88}$$

where m is the mass, $R(t)$ is the applied force, $r(t)$ is the force in the viscoelastic spring and $u(t)$ is the displacement of the mass. The constitutive equation for the spring can, e.g., be formulated in one of the two forms:

$$r(t) + b^{\alpha}\,\mathrm{D}^{\alpha}r(t) = k_{\infty}u(t) + kb^{\alpha}\,\mathrm{D}^{\alpha}u(t), \quad 0 < \alpha \leqslant 1 \tag{89a}$$

$$r(t) = k(u(t) - (f_{\alpha} * u)(t)) \tag{89b}$$

where k is the instantaneous stiffness and k_{∞} is the long time or relaxed stiffness in analogy with E and E_{∞}, respectively. $f_{\alpha}(t)$ is the memory kernel according to eqn (39) with $E = k$ and $E_{\infty} = k_{\infty}$. One way to formulate a single governing equation is to eliminate $r(t)$ by eqns (88) and (89a). If doing so, the composition rule for fractional derivatives must be employed. For the case of interest here, the composition rule takes the form, see Oldham and Spanier (1974)

$$\mathrm{D}^{\alpha}\,\mathrm{D}^{2}u(t) = \mathrm{D}^{2+\alpha}u(t) - \frac{t^{-\alpha-2}u(0)}{\Gamma(-1-\alpha)} - \frac{t^{-\alpha-1}\dot{u}(0)}{\Gamma(-\alpha)} \tag{90}$$

where $\dot{u}$ denotes one integer differentiation with respect to time. If homogeneous initial conditions $u(0) = \dot{u}(0) = 0$ are assumed the governing equation for the system might take the form

$$b^{\alpha}m\,\mathrm{D}^{2+\alpha}u(t) + m\,\mathrm{D}^{2}u(t) + kb^{\alpha}\,\mathrm{D}^{\alpha}u(t) + k_{\infty}u(t) = R(t) + b^{\alpha}\,\mathrm{D}^{\alpha}R(t) \tag{91}$$

This equation is generalized to a N-degree of freedom system, see Bagley and Torvik (1985), and the resulting finite element equation becomes

$$b^{\alpha}\mathbf{M}\,\mathrm{D}^{2+\alpha}\mathbf{u}(t) + \mathbf{M}\,\mathrm{D}^{2}\mathbf{u}(t) + b^{\alpha}\mathbf{K}\,\mathrm{D}^{\alpha}\mathbf{u}(t) + \mathbf{K}_{\infty}\mathbf{u}(t) = \mathbf{R}(t) + b^{\alpha}\,\mathrm{D}^{\alpha}\mathbf{R}^{t} \tag{92}$$

where $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{K}_{\infty}$ are $N \times N$ matrices, $\mathbf{R}(t)$ is the applied force vector and $\mathbf{u}(t)$ is the displacement vector. This form of discretized structural equation is used to solve transient problems in Bagley and Torvik (1985), Bagley and Calico (1991) and recently by Fenander (1996).

There are three things to notice about eqn (91) [and eqn (92)]. First, the use of the composition rule implies that $u(0) = \dot{u}(0) = 0$. Second, the order of the force equation eqn (88) has been raised and by use of the Laplace transform of the fractional order derivative operator it is realized that the formulation needs initial conditions on

$$\mathrm{D}^{\alpha-1}u(0), \quad u(0), \quad \mathrm{D}^{\alpha}u(0), \quad \mathrm{D}^{1}u(0), \quad \mathrm{D}^{\alpha+1}u(0) \tag{93}$$

The first quantity (93) is in fact a fractional integration and vanishes for any reasonable $u(t)$, see the definition of fractional integration in eqn (18). (Note, however, that e.g., $\mathrm{D}^{-1/2}[ct^{-1/2}]$ has non-vanishing limit as $t \to 0^{+}$.)

In any problem $u(0)$ and $\dot{u}(0)$ should be known but what about the conditions $\mathrm{D}^{\alpha}u(0)$ and $\mathrm{D}^{\alpha+1}u(0)$? It is tempting to prescribe homogeneous initial conditions but new difficulties will then be encountered. Consider a load $R(t)$ that is an eigenfunction to the operator D^{α} with eigenvalue $-1/b^{\alpha}$ [cf Section (6.4)]. The resulting differential equation is then homogeneous and together with homogeneous initial conditions it has only the trivial solution. This means that the part of an arbitrary load that is an "eigenfunction" will not cause any deformation, and in lack of a condition that excludes all loads which contain any eigenfunctional component one cannot be sure of having obtained the correct solution.

It might be argued that even for an eigenfunctional load, there will be a delta distribution term in the right hand side of eqn (91) so it is inhomogeneous at $t = 0$. This would require interpreting D^{α} as $\bar{\mathrm{D}}^{\alpha}$, the distributional derivative. However, if homogeneous conditions have been imposed on all of the quantities in eqn (93), and in particular $\mathrm{D}^{\alpha+1}(0)$, it is hard to see how the delta term on the right could be balanced by anything on the left.

Below we will discuss the problems connected to the eigenfunctional load, as well as with the impulsive load, in detail. However, before doing so, and to clarify the discussion above we refer to the case of $\alpha = 1$ in eqn (91), i.e., the spring is assumed to have a constitutive law of a Standard Linear Solid, which is discussed in Flügge (1975). The equation governing the motion, eqn (91), is then of order three and there is a need of initial conditions on $u(0)$, $\dot{u}(0)$ and $\ddot{u}(0)$. Again, the initial values of u and $\dot{u}$ should of course be known, but what about the acceleration $\ddot{u}$? Further, if a load that is proportional to $\mathrm{e}^{-t/b}$ is applied, the differential equation is homogeneous and in

combination with homogeneous initial conditions it is obvious that this load will not cause a deformation.

To avoid the difficulties described above we suggest to include the constitutive equation on convolution integral form rather than on differential form. By the use of eqns (88) and (89b) the equation governing the motion of the one-degree of freedom is formulated as

$$m\,\mathrm{D}^2u(t)+k(u(t)-(f_\alpha * u)(t)) = R(t) \tag{94}$$

This formulation requires initial conditions only on the physical quantities u and $\dot{u}$. Further, the formulation emphasizes that the whole past of the deformation must be taken into consideration when solving the equation. This can be generalized to a N-degree of freedom system in the same manner as eqn (92).

8.1. *The initial value problem*

Consider the initial value problems for the following three equations in the case of two simple loads:

$$m\,\mathrm{D}^2u(t)+k(u(t)-(f_\alpha * u)(t)) = R(t), \quad t>0 \tag{95a}$$

$$b^\alpha m\,\mathrm{D}^2[\mathrm{D}^\alpha u(t)]+m\,\mathrm{D}^2u(t)+kb^\alpha\,\mathrm{D}^\alpha u(t)+k_\infty u(t) = R(t)+b^\alpha\,\mathrm{D}^\alpha R(t), \quad t>0 \tag{95b}$$

$$b^\alpha m\,\mathrm{D}^{2+\alpha}u(t)+m\,\mathrm{D}^2u(t)+kb^\alpha\,\mathrm{D}^\alpha u(t)+k_\infty u(t) = R(t)+b^\alpha\,\mathrm{D}^\alpha R(t), \quad t>0 \tag{95c}$$

All three equations are of course intended to model the same physical system. Equation (95b) has been derived in the same manner as eqn (95c) but without making unrestricted use of the "naive" composition rule.

What are the initial condition appropriate for each of the above equations? Taking the Laplace transform of each equation indicates that the sets

$$(u(0^+), \mathrm{D}u(0^+)) \tag{96a}$$

$$(\mathrm{D}^{\alpha-1}u(0^+), u(0^+), \mathrm{D}u(0^+), \mathrm{D}^{\alpha-1}[\mathrm{D}^2u(0^+)]) \tag{96b}$$

$$(\mathrm{D}^{\alpha-1}u(0^+), u(0^+), \mathrm{D}u(0^+), \mathrm{D}^\alpha u(0^+), \mathrm{D}^{\alpha+1}u(0^+)) \tag{96c}$$

should be the appropriate ones for the respective equations [set (96a) for eqn (95a), etc]. Note that the set in eqn (96b) has one condition less than the set in eqn (96c).

It might be argued, as discussed in the previous section, that homogeneous initial conditions must be prescribed on all of the quantities in the sets in eqns (96a,b,c). An argument for this would be that any other conditions presuppose a previous displacement history. However, any values of at least the two physical initial conditions can be accomplished in an arbitrarily short time interval preceding $t=0$ by applying sufficiently large forces. Thus application of an impulsive load at $t=0$ is equivalent [at least as far as eqn (95a) is concerned] to specifying a non-vanishing value of $\mathrm{D}u(0^+)=v_0$ (but zero displacement) at $t=0^+$. Similarly the application of a delta function derivative load at $t=0$ is equivalent to specifying a non-vanishing value of $u(0^+)=u_0$ (but zero velocity) at $t=0^+$. A consequence of this is the freedom to specify any finite physical initial values $u(0^+)$ and $\mathrm{D}u(0^+)$, as they can be accomplished through combinations of those loads.

8.1.1. *Eigenfunctional load*

It is in fact straightforward to show that there even exist cases where a finite but non-zero initial condition on the highest order derivatives in eqns (96b) and (96c) is necessary. Consider the case of an eigenfunctional load, i.e.,

$$R(t) = \frac{C_0}{(1-k_\infty/k)} f_\alpha(t) \tag{97}$$

Here f_α is the same as in eqn (39) but with $E = k$ and $E_\infty = k_\infty$. The right hand sides of eqns (95b) and (95c) vanish for this load, and homogeneous initial conditions on all of the quantities of eqns (96b) and (96c) would yield only the trivial solution. However, the right hand side of eqn (95a) does not vanish for this load. To be specific, consider the case of $\alpha = 1/2$ and assume homogeneous initial conditions on the physical quantities of eqn (96a). A straightforward application of the Laplace transform to eqn (95a) yields the solution

$$u(s) = u_1(s) = \frac{C_0}{ms^2(1+(bs)^{1/2}) + k_\infty + k(bs)^{1/2}} \tag{98}$$

This solution can be expanded in an asymptotic series as $s \to \infty$

$$u_1(s) \sim \frac{C_0}{mb^{1/2}s^{5/2}} - \frac{C_0}{mbs^3} + \frac{C_0}{mb^{3/2}s^{7/2}} + \mathcal{O}\left(\frac{1}{s^4}\right) \tag{99}$$

A term-wise Laplace inversion yields the first few terms in a series expansion of $u_1(t)$ for small times as

$$u_1(t) = \frac{4C_0 t(t/b)^{1/2}}{3\sqrt{\pi}m} - \frac{C_0 t(t/b)}{2m} + \frac{8C_0 t(t/b)^{3/2}}{15\sqrt{\pi}m} + \mathcal{O}(t^3) \tag{100}$$

which satisfies the initial conditions:

$$\mathrm{D}^{1/2-1}u_1(0^+) = u(0^+) = \mathrm{D}^{1/2}u_1(0^+) = \mathrm{D}u_1(0^+) = 0 \tag{101a}$$

and

$$\mathrm{D}^{1+1/2}u_1(0^+) = \frac{C_0}{b^{1/2}m} \tag{101b}$$

In this case also

$$\mathrm{D}^{1/2-1}[\mathrm{D}^2 u_1(0^+)] = \frac{C_0}{b^{1/2}m} \tag{101c}$$

8.1.2. *Impulsive load*

Consider the case of an impulsive load applied at time $t = 0$ ($R(t) = F_0\delta(t)$). As far as eqn (95a) goes this is equivalent to having a zero load and specifying a non-vanishing initial velocity, but zero displacement at $t = 0^+$, and the data of the problem under consideration can be formulated as

$$R(t) = 0, \quad t > 0$$
$$u(0^+) = 0$$
$$Du(0^+) = v_0 = \frac{F_0}{m} \tag{102}$$

A straightforward application of the Laplace transform to eqn (95a) again yields a solution which can be expanded for small times as

$$u_2(t) = v_0 t - \frac{v_0 k t^3}{6m} + \frac{16 v_0 (k - k_\infty) t^3 (t/b)^{1/2}}{105 b m \sqrt{\pi}} + \mathcal{O}(t^4) \tag{103}$$

What are the initial conditions corresponding to this solution? The unrestricted use of the composition rule when deriving eqn (95b) in this case entailes initial conditions which cannot be met by the displacement due to an impulsive load. Of the set in eqn (96c) we have from eqn (103) that

$$\mathrm{D}^{1/2-1} u_2(0^+) = u_2(0^+) = \mathrm{D}^{1/2} u_2(0^+) = 0, \quad \mathrm{D} u_2(0^+) = v_0 \tag{104}$$

However, and this is the interesting point, we find that

$$\mathrm{D}^{1+1/2} u_2(0^+) = \infty \tag{105}$$

so the set in eqn (96c) is in a sense incompatible with the displacement due to an impulsive load. If eqn (95c) with a derivative of the form $\mathrm{D}^{\alpha}[\mathrm{D}^2]$ had been used instead, this problem would not have appeared. This is so because the required highest order initial condition in eqn (96b) turns out to be finite and in fact vanishes:

$$\mathrm{D}^{1/2-1}[\mathrm{D}^2 u_2(0^+)] = 0 \tag{106}$$

The use of either of the two formulations eqns (95b) and (95c) also calls for some delicacy in the use of the Laplace transform: there is, even for the case of the eigenfunction load, reason to consider carefully whether the Laplace transform is to be taken from $t = 0^+$ (as done here) or from $t = 0^-$ (which would require initial conditions at $t = 0^-$), and correspondingly whether the fractional derivative is $\mathrm{D} = {}_{0^-}\mathrm{D}$ or $\mathrm{D} = {}_{0^+}\mathrm{D}$. This has implications for the presence or absence of unbalanced singular terms in the two eqns (95b) and (95c). In the case of eqn (95a) none of these questions are at all problematic.

We can thus conclude the following: eqn (95a) is the simplest to use when considering initial value problems for a fractional calculus viscoelastic system, since it not only requires only two initial conditions, but these are the "physical" ones, displacement and velocity. Equations (95b) and (95c) on the other hand require more conditions, and these are conditions on "unphysical" quantities. The simple remedy of making all such conditions homogeneous is not available, as seen from the two examples above, and in the case of the impulsive load the set of initial conditions for eqn (95b) requires one initial value to be infinite, an awkward situation to say the least.

In the case of eqn (95a) we have the freedom to consider certain displacements as arising either from initial conditions or from the application of concentrated loads, the same freedom is restricted in the case of eqns (95b) and (95c). In the case of the latter two equations one must carefully consider whether the convolution integrals defining the fractional derivatives are to be taken from

$t = 0^+$ or $t = 0^-$, and similarly for the Laplace integrals, if solving by means of such transform, as all this has implications for the singularities of the equations as well as for the choice of initial conditions.

9. Computational algorithm

In this section we present a time discretization scheme for solving the structural equation for transient loaded fractional viscoelastic spring-mass system. A governing equation involving the convolution integral constitutive relation, see eqn (94), is used.

The basic idea is to use the expression for the fractional order derivative of the convolution term, $\mathrm{D}^{\alpha}(f_{\alpha} * u)(t)$, derived in Section (6.4), eqn (56) in connection with Newmark's method for time discretization. The numerical differentiation of fractional order is approximated by use of a suitable truncation of the Grünwald algorithm for differ integration (i.e. fractional differentiation or fractional integration) as, see Oldham and Spanier (1974),

$$[\mathrm{D}^{\alpha}(f_{\alpha} * u)]_{n+1} = \frac{1}{\Delta t^{\alpha}}\left[(f_{\alpha} * u)_{n+1} + \sum_{j=1}^{n} B_j(\alpha)(f_{\alpha} * u)_{n+1-j}\right] \tag{107}$$

with

$$B_j(\alpha) = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} \tag{108}$$

It is then assumed that the spacing in time is uniform (i.e. $(f_{\alpha} * u)_n = (f_{\alpha} * u)(n\Delta t)$). This approximation is accurate also for small n as long as the time increment Δt is small enough. There are several other algorithms for numerical fractional differentiation, Oldham and Spanier (1974); one of these might be more efficient than eqn (107). The efficiency of these different algorithms will not be investigated at present. With the application of the Backward Euler rule (in time) to the expression for the fractional derivative of the convolution [eqn (56)] together with approximation of the fractional derivative of the convolution term [eqn (107)], we obtain an approximation of the convolution term as

$$(f_{\alpha} * u)_{n+1} = \frac{(\Delta t)^{\alpha}}{(\Delta t)^{\alpha} + b^{\alpha}}\left[\frac{k - k_{\infty}}{k} u_{n+1} - \left(\frac{b}{\Delta t}\right)^{\alpha} \sum_{j=1}^{n} B_j(\alpha)(f_{\alpha} * u)_{n+1-j}\right] \tag{109}$$

with the initial value $(f_{\alpha} * u)(0) = 0$. The calculations of the ratios between gamma functions are simplified by use of the recursion formula:

$$\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} = \frac{(j-1-\alpha)}{j}\frac{\Gamma(j-1-\alpha)}{\Gamma(j)} \tag{110}$$

From the expression, eqn (109), above it is seen that the complete history of the convolution term should be saved and included in the sum in each time step. In reality, the sum must of course be truncated. One thing that is worth noticing is that no explicit evaluation of the memory kernel is needed, which is a desirable feature.

9.1. *Structural response*

Newmark's scheme is one of the simplest and most popular schemes for time integration of the structural response. It has here used due to its simplicity. The displacement and first order time derivative are expanded as (see Cook et al., 1989)

$$u_{n+1} = u_n + \Delta t u_n + \frac{(\Delta t)^2}{2}[(1-2\beta)\ddot{u}_n + 2\beta\ddot{u}_{n+1}] \tag{111a}$$

and the nodal velocities from

$$\dot{u}_{n+1} = \dot{u}_n + \Delta t[(1-\gamma)\ddot{u}_n + \gamma\ddot{u}_{n+1}] \tag{111b}$$

where β and γ are chosen to control stability and accuracy. This together with the discretized dynamic equation (evaluated at the end of each time step),

$$m\ddot{u}_{n+1} + k[u_{n+1} - (f_\alpha * u)_{n+1}] = R_{n+1} \tag{112}$$

and initial conditions on $u(0)$, and $\dot{u}(0)$, and the discretized expression for the convolution term $(f_\alpha * u)$ [eqn (109)] makes it possible to calculate the four unknowns: u_{n+1}, $\dot{u}_{n+1}$, $\ddot{u}_{n+1}$ and $(f_\alpha * u)_{n+1}$. A suitable scheme is as follows:

- Calculate $\ddot{u}(0)$ from the dynamic equation, eqn (112).
- By combining eqns (111a), (112) and (109), u_{n+1} is solved from the expression:

$$u_{n+1} = A\left(\mathbf{d}_n + \Delta t\dot{u}_n + \frac{(\Delta t)^2}{2}(1-2\beta)\ddot{u}_n + (\Delta t)^2\beta m^{-1}R_{n+1} - \frac{(\Delta t)^2\beta b^\alpha}{(\Delta t)^\alpha + b^\alpha} m^{-1}k\sum_{j=1}^{n} B_j(\alpha)(f * u)_{n+1-j}\right) \tag{113}$$

with

$$A = \left[1 + (\Delta t)^2\beta m^{-1}\left(k - \frac{(\Delta t)^\alpha}{(\Delta t)^\alpha + b^\alpha}(k - k_\infty)\right)\right]^{-1}$$

- Solve $(f_\alpha * u)_{n+1}$ from eqn (109).
- Solve $\ddot{u}_{n+1}$ from eqn (112).
- Solve $\dot{u}_{n+1}$ from eqn (111b).

In the undamped case the stability conditions can be found in, e.g., Cook et al. (1989).

Letting $\beta = 0$, which is referred to as "nearly" explicit in Newmark's algorithm, produces a very simple scheme to calculate the four unknowns:

- Calculate $\ddot{u}(0)$ from the dynamic equation, eqn (112).
- Solve u_{n+1} from eqn (111a).
- Solve $(f_\alpha * u)_{n+1}$ from eqn (109).
- Solve $\ddot{u}_{n+1}$ from eqn (112).
- Solve $\dot{u}_{n+1}$ from eqn (111b).

In the undamped case and $\beta = 0$ the algorithm is known to be conditionally stable for $\gamma \geqslant 1/2$ and the critical time step is

$$\Delta t_{\text{crit}} = \frac{1}{\omega_{\max}\sqrt{\gamma/2}} \tag{114}$$

here is $\omega_{\max}$ the (highest) undamped natural frequency. For the one-degree of freedom $\omega_{\max} = \sqrt{k/m}$. This critical time step is conservative for the viscoelastic structure since the viscoelastic spring weakens with increasing time which results in decreasing $\omega_{\max}$. Stability and possible faster algorithms such as multi-steps methods will not be discussed further at present.

9.2. *Numerical examples*

In this section a few numerical examples for a transiently loaded viscoelastic one-degree of freedom system are presented to demonstrate the algorithm described above. A "nearly" explicit scheme with $\beta = 0$ and $\gamma = 1/2$ is used. The time step chosen are sufficiently smaller than the critical time step in the undamped case [eqn (114)]. Numerical values used for the one-degree of freedom system under consideration are $m = 1$ kg, $k = 2$ Nm and $k_\infty/k = 1/2$. Different values of the relaxation times (b) are used.

Consider first the case of $\alpha = 1/2$ and $b = 1$ s which enables us to compare the numerical results obtained by the algorithm presented above and the time series expansions of the analytical solutions according to eqns (100) and (103) for the cases of eigenfunctional loading and impulsive loading, respectively. Figure 4 shows the numerically obtained displacement solution and the time series expansion in the case of eigenfunctional load with amplitude $C_0 = 1$ Ns. In Fig. 5, the numerically obtained displacement solution and the time series expansion in the case of unit impulsive loading are displayed. As seen in Figs 4 and 5, the agreement between the numerical solutions and the time

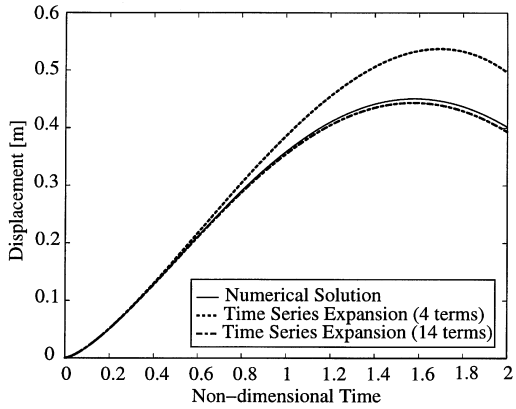

Fig. 4. Displacement vs non-dimensional time t/b for viscoelastic one-degree of freedom system (with $\alpha = 1/2$) subjected to a unit eigenfunctional load at time $t = 0$. Dotted and dash–dotted line represent time series expansions of analytic solution using three terms and nine terms in the series, respectively. Time series are valid for short times.

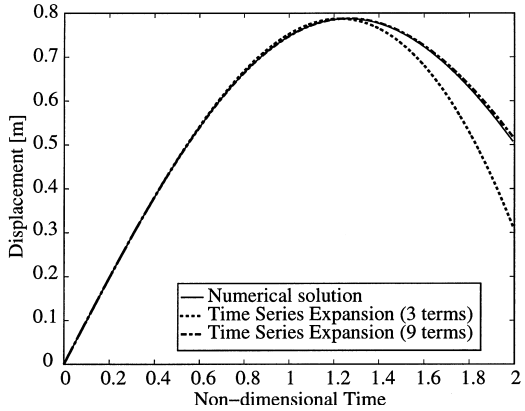

Fig. 5. Displacement vs non-dimensional time t/b viscoelastic one-degree of freedom system (with $\alpha = 1/2$) subjected to a unit impulse load at time $t = 0$. Dotted and dash–dotted line represent time series expansion of analytic solution using three terms and nine terms in the series, respectively. Time series are valid for short times.

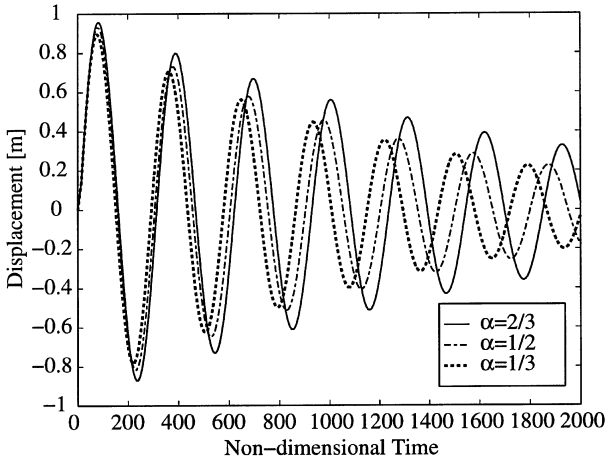

Fig. 6. Displacement vs non-dimensional time t/b for viscoelastic one-degree of freedom system subjected to a unit impulse load at time $t = 0$. The influence of different values of the fractional derivative exponent α is shown.

series expansions are good for short times, indicating that the algorithm is accurate. For longer times, the solutions deviate because the time series expansions are no longer valid.

In Figs 6 and 7 we use different values of the fractional derivative exponent $\alpha \in (0, 1)$ and $b = 0.02$ s. Figure 6 shows numerical displacement solution in the case of a unit impulse load applied at time $t = 0$. As seen in Fig. 6, oscillations are more damped with increasing value of the fractional derivative exponent. Figure 7 shows normalized displacement solution in the case of a unit step load applied at time $t = 0$, displacement is normalized with the quasi-static long time displacement (i.e., $u_{\text{stat}} = R/k = 1$ m).

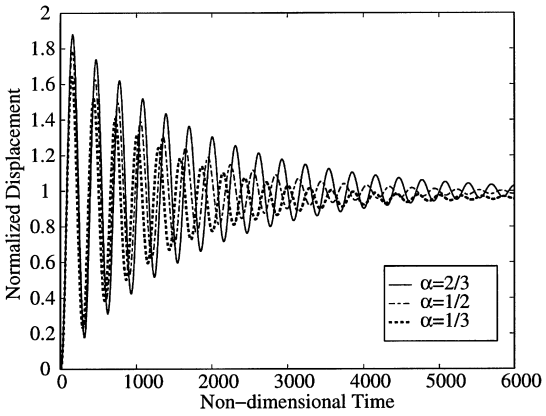

Fig. 7. Normalized displacement u/u_{stat} vs non-dimensional time t/b for viscoelastic one-degree of freedom system subjected to a unit step load at time $t = 0$. The influence of different values of the fractional derivative exponent α is shown.

10. Conclusions

In the present paper we have discussed some of the restrictions that must be put on material or constitutive models to describe damping. There are, as we have emphasized, a number of pitfalls which must be avoided, in particular if one insists in doing the modeling in the frequency domain. Some properties which are easy to verify in the time domain, become concealed in the frequency domain and much harder to recognize. Our conclusion is that it is essential to keep a time domain perspective on the modeling of damping.

Some damping models have been commented on in more detail. Regarding the Standard Linear Solid model, we have emphasized that it is equivalent to two other formulations. The other model we have focused on, the fractional derivative damping model, exhibits a number of interesting features. Also for this model we have pointed out that is equivalent to at least two other formulations, including an internal variable model. Emphasis has been put on the properties of the memory kernel corresponding to the fractional derivative model. An outcome of this analysis is a result on the time evolution of convolutions involving the memory kernel. For applications to structural dynamics a time discretization of this result is shown to be potentially very useful. Regarding the higher order equations that have been used in connection with structural dynamics applications of the fractional derivative model we have pointed out some difficulties.

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